eg) M = the set of all 2×2 matrices $\longrightarrow$ it's a vector space $\equiv V$

H = the set of symmetric $(A=A^T)$ 2×2 matrices $\Rightarrow H \in V$

a) zero matrix $(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in H$

b) let $A_1, A_2 \in H$ ie: $A_1^T = A_1$, $A_2^T = A_2$

$(A_1 + A_2)^T = A_1^T + A_2^T = A_1 + A_2 \in H$

c) C = scalar $A = A^T$

$(CA)^T = C(A)^T = CA \in H$

Ex. Which of the following 2 subsets is a subspace of $\mathbb{R}^2$?

a) The set of all points on the line $x+2y=0$

b) The set of all points on the line $x+2y$

a) 1- $0 = (0,0) \in H$

2- let $y = t \Rightarrow x = -2t$ $(-2t, t)$

$V_1 = (-2t_1, t_1)$, $V_2 = (-2t_2, t_2)$

$V_1 + V_2 = (-2(t_1+t_2), (t_1+t_2))$

$= (-2t, t)$ where $t = t_1 + t_2$

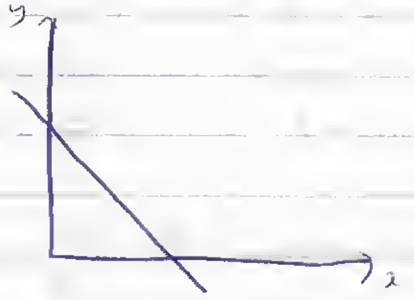
3- C = scalar , $CV = (-2ct, ct) = c(-2t, t)$

$\therefore$ H is a subspace of $\mathbb{R}^2$

b) let $H: x + 2y = 1$

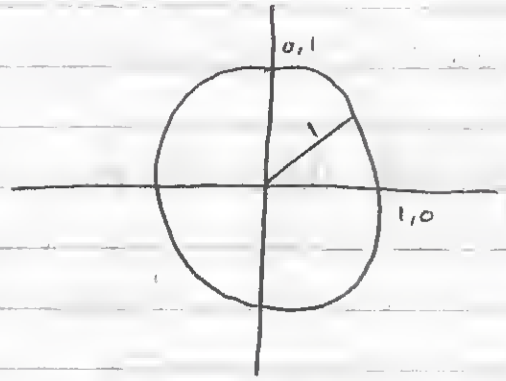
$\therefore 0\ (0,0) \notin H$

H is not a subspace



Ex let H be the set of all points on the unit circle $x^2+y^2=1$, is H a subset of $\mathbb{R}^2$?

H is not a subspace of $\mathbb{R}^2$ because
1- $\underline{0} = (0,0) \notin H$
2- not closed under addition



Ex The set of singular matrices of $M_{n\times n}$. All square matrices are vector spaces.

A singular matrix $\Rightarrow$ $|A| = 0$

$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$|A_1| = 0$, $|A_2| = 0$ $|A_1 + A_2| = 0$

$A_1 + A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq 0$ H is not closed under addition
$\Rightarrow$ H is not a subspace of $M_{n\times n}$

<u>Linear combination and linear indepence</u>

Def: A vector u in a vector space V is called a linear combinations of vectors $u_1, u_2, u_3 \cdots u_n$ in V if u can be written in the form
$u = c_1u_1 + c_2u_2 + c_3u_3 + \cdots c_nu_n$ where $c_1, c_2, c_3 \cdots c_n$ are scalars

Ex $S = \{ \underset{u_1}{(1,3,1)}\ \underset{u_2}{(0,1,2)}\ \underset{u_3}{(1,0,-5)}$

Show that u_1 is a linear combination of u_2, u_3

$u_1 = c_1 u_2 + c_2 u_3$

$u_1 = 3u_2 + u_3$

$\therefore$ u_1 is a linear combination of u_2, u_3

Write (if possible) the vector $w = (1,1,1)$ as a linear combination of the vectors $u_1 = (1,2,3)$

$u_2 = (0,1,2)$

$u_3 = (-1,0,1)$

It's required to find c_1, c_2, c_3 such that

$(1,1,1) = c_1(1,2,3) + c_2(0,1,2) + c_3(-1,0,1)$

$$c_1 + 0 - c_3 = 1$$
$$2c_1 + c_2 + 0 = 1$$
$$3c_1 + 2c_2 + c_3 = 1$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right) \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & -2 \end{array}\right) \xrightarrow{R_3 - 2R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$c_3 = t$

$c_2 + 2t = -1$, $c_2 = -1 - 2t$

$c_1 - t = 1$ $\quad c_1 = 1 + t$

let $t = 2$

$c_1 = 3$, $c_2 = -5$, $c_3 = 2$

Ex. If possible, write the vector $w=(1,-2,2)$ as a linear combination of vectors $V_1=(-1,0,-1)$ $V_2=(2,1,0)$ $V_3(3,2,1)$

$$w = C_1V_1 + C_2V_2 + C_3V_3$$

$$(1,-2,2) = C_1(1,0,-1) + C_2(2,1,0) + C_3(3,2,1)$$

$$\begin{aligned} 1 &= C_1 + 2C_2 + 3C_3 \\ -2 &= C_2 + 2C_3 \\ 2 &= -C_1 + C_3 \end{aligned} \Longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 7 \end{array}\right) \text{(after solving)}$$

matrix not consistent $\Rightarrow$ w is not a linear combination

Linear dependence and independence

A set of vectors $S=\{V_1, V_2, \dots V_n\}$ in a vector space V is called linearly independent if the vector equation $C_1V_1 + C_2V_2 + \dots + C_nV_n = 0$ has only the trivial solution $C_1 = C_2 = C_n = 0$.

Ex. Determine whether the set of vectors in $\mathbb{R}^3$ is linearly independent or not.

$V_1=(1,2,3)$ $V_2=(0,1,2)$ $V_3=(-2,0,1)$

$$C_1V_1 + C_2V_2 + C_3V_3 = 0$$

$$C_1(1,2,3) + C_2(0,1,2) + C_3(-2,0,1)$$

$$\begin{aligned} C_1 \quad\quad - 2C_3 &= 0 \\ 2C_1 + C_2 \quad\quad &= 0 \\ 3C_1 + 2C_2 + C_3 &= 0 \end{aligned} \xrightarrow[\text{Solution}]{\text{after}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right)$$

$C_1 = C_2 = C_3 = 0$ $\therefore$ V_1, V_2, V_3 are linearly independent

Ex. Determine whether the set of vectors $V_1=\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$, $V_2=\begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}$, $V_3=\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$ are linearly independent or not

$$C_1V_1 + C_2V_2 + C_3V_3 = 0$$

$$C_1\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} + C_2\begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix} + C_3\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$2C_1 + 3C_2 + C_3 = 0$ $\quad$ $C_1 + C_2 = 0$

$C_1 = 0$ $\quad$ $C_1 = 0$, $C_2 = 0$, $C_3 = 0$ $\therefore$ V_1, V_2, V_3 are linearly independent

$\cancel{2C_2}\ 2C_2 + 2C_3 = 0$

Theorem: Two vectors u, v are linearly dependent if and only if one is a scalar multiple of the other

properties of vector spaces

[1] Spamming / Generating set of a vector space

Def. Let $S = \{V_1, V_2, --- V_n\}$ be a subset of a vector space V.
Then set S is called spamming set of V if every vector in V can be written as a linear combination of vectors in S.

Eg. the set S $\{(1,0,0), (0,1,0), (0,0,1)\}$ is the spamming set of $\mathbb{R}^3$

$(5,8,7) = 5(1,0,0) + 8(0,1,0) + 7(0,0,1)$

Eg. The set S $\{1, x, x^2\}$ is the spamming set of P_2

P_2 = the set of all polynomials of degree ≤ 2

$P(x) = 8 = 8(1) + 0(x) + 0(x^2)$

$Q(x) = 5x + 3 = 3(1) + 5(x) + 0(x^2)$

$R(x) = 6x^2 - 3x - 2 = -2(1) + (-3)(x) + 6(x^2)$

[2] Basis of a vector space

Def. A set of vectors $S = \{V_1, V_2, ----, V_n\}$ in a vector space V is called a basis if the following conditions are true

a) S spams V

b) S is ~~entirely~~ linearly independent

Ex. Show that the set $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis of $\mathbb{R}^2$

a) S spams $\mathbb{R}^2$

b) $C_1(1,0,0) + C_2(0,1,0) + C_3(0,0,1) = 0$

$C_1 = C_2 = C_3 \Rightarrow$ linearly independent

$\therefore$ S is a basis of $\mathbb{R}^3$